Differential Geometry of Stable Vector Bundles

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- Weil (1938) began the generalization of divisor theory to that of matrix divisors which corresponds to the vector bundles. The classification of vector bundles of rank $n > 1$ is much harder than for line bundles partly because there is no group structure.

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- Grothendieck (1956) showed that for genus 0 the classification is trivial in the sense that every holomorphic vector bundle over $\mathbb{P}^1$ is a direct sum of line bundles (a result known in a different language to Hilbert, Plemelj and Birkhoff, and prior to them to Dedekind and Weber).

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- Atiyah (1957) classified all vector bundles over an elliptic curve and made some remarks concerning vector bundles over curves of higher genus.
- In general in order to get a good moduli space one has to restrict to the class of stable bundles as introduced by Mumford, otherwise one gets non-Hausdorff phenomena.

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- Soon after Mumford , Narasimhan and Seshadri (1965) related the notion of stability to the existence of a unitary flat structure (in the case of trivial determinant) or equivalently a flat connection compatible with an appropriate Hermitian metric.
- In fact the major breakthrough come with the discovery of Narasimhan and Seshadri that bundles are stable if and they arise from the irreducible (projective)unitary representation of the fundamental group.

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- Schwarzenberger showed that every rank 2 vector bundle on a smooth surface X is of the form $\pi_* L$, where $\pi : \widetilde{X} \longrightarrow X$ is a smooth double cover of X and L is a line bundle on $\widetilde{X}$. In fact the direct image $\pi_* L$ is a rank two vector bundle over X.

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- He then applied this construction to construct bundles on $\mathbb{P}^2$ which were not almost decomposable $(\dim H^0(X, \mathrm{End}(E)) = 1)$; these turn out to be exactly the stable bundles on $\mathbb{P}^2$.

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- He showed further that, if V is a stable rank 2 vector bundle on $\mathbb{P}^2$, then the Chern classes for V satisfy the basic inequality $c_1(V)^2 \leq 4c_2(V)$.

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- Takemoto (1972, 1973) gave the straight forward generalization to higher-dimensional (polarized) smooth projective varieties that we have simply called stability here (this definition is also called Mumford-Takemoto stability, μ-stability, or slope stability).
- Aside from proving boundedness results for surfaces, he was unable to prove the existence of a moduli space with this definition (and in fact it is still an open question whether the set of all semistable bundles forms a moduli space in a natural way.

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- This result was generalized by Makuyama (1978) to the case where X has arbitrary dimension. The differential geometric meaning of Mumford stability is the Kobayashi-Hitcbin structure, that every stable vector bundle has a Hermitian-Einstein connection, unique in an appropriate sense.

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- The easier converse, that an irreducible Hermitian-Einstein connection defines a holomorphic structure for which the bundle is stable, was established previously by Kobayashi and Lubke.

Brief History

- In summary:
  In order to construct a good moduli spaces for vector bundles over algebraic curves, Mumford introduced the concept of a stable vector bundle. This concept has been generalized to vector bundles and, more generally, coherent sheaves over algebraic manifolds by Takemoto, Bogomolov and Gieseker. The differential geometric counterpart to the stability, is the concept of an Einstein- Hermitian vector bundle.

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- In fact Narasimhan and Seshadri proved that the Stable holomorphic vector bundles over a compact Riemann surface are precisely those arising from irreducible projective unitary representations of the fundamental group.
- In this lecture I want to introduce Donaldson method which gives different, more direct, proof of this fact using the Differential Geometry of Connections on holomorphic bundles.
- The idea of this method essentially goes back to the famous paper written by Atiyah and Bott (Yang-Mills equation over Riemann surfaces, 1983) in which the result of Narasimhan and Seshadri is used to calculate the cohomology of moduli spaces of stable bundles.

Mumford Stability

- Let M be a compact Riemann surface with a Hermitian metric. If E is a vector bundle over M define:

$$\mu(E) = \text{degree}(E)/\text{rank}(E),$$

where the degree is the Chern class of E. A holomorphic bundle $\mathcal{E}$ is defined to be stable if for all proper holomorphic sub-bundles $\mathcal{F} < \mathcal{E}$ we have :

$$\mu(\mathcal{F}) < \mu(\mathcal{E})$$

The main theorem

- The Main Theorem. An indecomposable holomorphic bundle $\mathcal{E}$ over M is stable if and only if there is a unitary connection on $\mathcal{E}$ having constant central curvature $*F = -2\pi i\mu(\mathcal{E})$. Such a connection is unique up to isomorphism.

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- The Main Theorem. An indecomposable holomorphic bundle $\mathcal{E}$ over M is stable if and only if there is a unitary connection on $\mathcal{E}$ having constant central curvature $*F = -2\pi i \mu(\mathcal{E})$. Such a connection is unique up to isomorphism.
- Note. If $\deg(\mathcal{E}) = 0$, these connections are flat and it can be easily shown that they are given by unitary representations of the fundamental group. In the general case it is also easy to prove the equivalence of this form of the result with the statement of Narasimhan and Seshadri (Atiyah and Bott).

Connections on Complex Vector Bundles

- Let M be a general manifold and let E be a C^∞ complex vector bundle on M of rank r. Recall that a connection on E is a C-linear map D from C^∞ sections of E to sections of $A^1(E) = E \otimes A^1(M)$, where $A^1(M)$ is the C^∞ 1-forms on M satisfying the Leibnitz rule:

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- For all sections s of E and C^∞ functions f on M,

$$D(fs) = fDs + s \otimes df$$

It follows that the difference of two connections is a C^∞ 1-form with coefficients in $\mathrm{End}E$, and in fact the space of all connections is an affine space for $A^1(\mathrm{End}E)$.

Connections on Complex Vector Bundles

- There is a natural extension of D to an operator from $A^p(E) \longrightarrow A^{p+1}(E)$, where $A^p(E)$ is the vector bundle of p-forms with coefficients in E, by requiring the graded Leibniz rule:

$$D(\phi \otimes s) = d\phi \otimes s + (-)^p \phi \otimes Ds$$

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- Using this extension we define the curvature R of D to be

$$R = DoD : A^0(E) \longrightarrow A^2(E).$$

It can be easily checked that R is A^0-linear. Hence, R is a 2-form on M with values in $\mathrm{End}(E)$ or equivalently R is a C^∞ section of $A^2(\mathrm{End}(E)) = A^2(M) \otimes \mathrm{End(E)}$.

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- The vector bundle E (or more precisely the pair (E, D)) is flat if $D^2 = 0$. As a corollary of the Frobenius theorem, if E is flat and M is simply connected, then E is trivialized by global sections $s_1, ..., s_r$ such that $Dsi = 0$ for all i.

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- More precisely, if E is a vector bundle with a flat connection D. Let x_0 be a point of M and π_1 the fundamental group of M with reference point x_0. Since the connection is flat, the parallel displacement along a closed curve γ starting at x_0 depends only on the homotopy class of γ.

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- So the parallel displacement gives rise to a representation

$$\rho : \pi_1(M, *) \longrightarrow GL(r, C)$$

The image of ρ is the so called holonomy group of D.

Connections on Complex Vector Bundles

- Conversely, given a representation $\rho : \pi_1(M, *) \longrightarrow GL(r, C)$, we can construct a flat vector bundle E by setting

$$E = \widetilde{M} \times_\rho C^r$$

, where $\widetilde{M}$ is the universal covering of M and $\widetilde{M} \times_\rho C^r$ denotes the quotient of $\widetilde{M} \times C^r$ by the action of π_1 given by

$$\gamma : (x, v) \in \widetilde{M} \times C^r \longmapsto (\gamma(x), \rho(\gamma)v) \in \widetilde{M} \times C^r$$

The vector bundle defined by above is said to be defined by the representation ρ.

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  2. E is defined by a representation $\rho : \pi_1 \longrightarrow GL(r, C)$.
- Similarly a connection D on a vector bundle E over M is called projectively flat if the curvature $R = DoD : A^0(E) \longrightarrow A^2(E)$ which we saw that it can be regarded as an element in $A^2(\mathrm{End}(E)) = A^2(M) \otimes \mathrm{End}(E)$ has the form $R = \alpha I_E$ where α is a 2-form on M and I_E is the identity map in the group $\mathrm{End}(E)$.

Connections on Complex Vector Bundles

- In the cases of interest, E will have a Hermitian metric $\langle ., . \rangle$, and D will be compatible with the metric in the sense that

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- If s_i is an orthonormal basis with respect to the inner product, then the connection matrix A is skew-Hermitian, or in other words it lies in the Lie algebra $u(r)$ of the unitary group $U(r)$. We say that the connection A is unitary or Hermitian.

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- In this case, the curvature, computed in a local orthonormal frame, is a skew-Hermitian matrix of 2-forms. The flat vector bundles E whose connections are compatible with a Hermitian metric essentially correspond to representations of $\pi_1(M, *)$ into $U(r)$ rather than into $GL(r, C)$.

Chern Class

- If E is a Hermitian vector bundle and D is a connection which is compatible with the metric on E, then we can consider the characteristic polynomial:

$$\det(\frac{i}{2\pi}D^2 + t\mathrm{id}) = \sum_{k=0}^{r} c_k(E)t^{r-k}$$

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- Here the coefficients $c_k(E)$ turn out to be closed forms of degree $2k$ representing the Chern classes of the vector bundle E.
- For example, $c_1(E) = (i/2\pi)\mathrm{Tr}(D^2)$. Note that, if D is flat, then $c_i(E) = 0$ for all $i > 0$.

Chern Class

- suppose that M is a complex manifold (in our case of discussion Riemann surface), so that $d = \partial + \overline{\partial}$. Let $\Omega^{p,q}(M)$ be the vector bundle of forms of type (p, q), and, for a complex vector bundle E, define $\Omega^{p,q}(E)$ similarly.

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- If E is holomorphic, then $\overline{\partial}$ is well defined on C^∞ sections of E, and we say that the connection D is compatible with the complex structure, if

$$\pi^{0,1}(D) = \overline{\partial}$$

, where $\pi^{0,1} : A^1(E) \longrightarrow \Omega^{0,1}(E)$ is the projection induced from the projection of the l-forms on M to the, $(0,l)$-forms.

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- In this case $\pi^{0,2}(D^2) = 0$, in other words, the curvature has no component of type $(0, 2)$.

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- Conversely, if E is a C^∞ vector bundle and D is a connection on E such that $\pi^{0,2}(D^2) = 0$, then there exists a unique holomorphic structure on E for which D is a compatible connection.

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- Every holomorphic vector bundle E with a Hermitian metric has a unique unitary connection D which is compatible with the complex structure. It is referred to D as the compatible unitary connection associated to the metric.
- In this case, since $\overline{\partial}^2 = 0$, D^2 has no component of type $(0, 2)$, and since it is skew-Hermitian, it has no $(2, 0)$-component either. Thus, the curvature D^2 lives in $\Omega^{1,1}$. It follows that the Chern classes $C_k(E)$ are represented by real forms of type (k, k).

Connections on Complex Vector Bundles

- Definition. If E is a C^∞ vector bundle over a Riemann surface X a unitary connection A on E gives an operator $d_A : \Omega^0(E) \longrightarrow \Omega^1(E)$ which has a $(0,1)$ component $\overline{\partial}_A : \Omega^0(E) \longrightarrow \Omega^{0,1}(E)$ and this defines a holomorphic structure $\mathcal{E}_A$ on E (Because according to a theorem By Atiyah and Bott there are sufficiently many local solutions of the elliptic equation $\overline{\partial}_A(s) = 0$).

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- Conversely if $\mathcal{E}$ is a Holomorphic structure on E there is a unique way to define a unitary connection A such that $\mathcal{E} = \mathcal{E}_A$. So there is one to one correspondence between unitary connections on E and holomorphic structure on E

Gauge Group

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- A Connection on E induces a connection on all associated bundles, in particular, on the bundle of Endomorphisms End E.
- The gauge group $\mathcal{G}$ of unitary automorphisms of E acts as a symmetry group on the affine space $\mathcal{A}$ of all unitary connection on E by: $u(A) = A - d_A u u^{-1}$, $u \in \mathcal{G}$ and $A \in \mathcal{A}$.

Gauge Group

- The action also extends the complexification $\mathcal{G}^C =$ group of general linear automorphisms of E. Connections define Isomorphic holomorphic structure precisely when they lie in the same $\mathcal{G}^C$ orbit. So the set of $\mathcal{G}^C$ orbits parametrize all the holomorphic bundles of the same degree and rank as E (there are no further topological invariants of bundles over X).

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- $F(A + a) = F(A) + d_A a + a \wedge a$.
- The plan of the proof. The case of line bundles is an easy consequence of the Hodge theory . Suppose inductively that the result has been proved for bundles of lower rank, then we choose a minimizing sequence in $O(\mathcal{E})$ for a carefully constructed functional J in terms of the curvature and extract a weakly convergent subsequence.

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  2. The limiting connection is in another orbit $O(\mathcal{F})$ and we deduce that $\mathcal{E}$ is not stable, a contradiction.

The Yang-Mills Functional

- Definition of the functional J. The trace norm is defined on $n \times n$ Hermitian matrices by

$$\nu(X) = \mathrm{Tr}(X^* X)^{1/2} = \sum_{i=1}^{n} |\lambda_i|,$$

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$$\nu(X) = \mathrm{Tr}(X^* X)^{1/2} = \sum_{i=1}^{n} |\lambda_i|,$$

- where $\{\lambda_i\}$ are the eigenvalues of X. Applying ν in each fiber we define, for any smooth self-adjoint section s in $\Omega^0(\mathrm{EndE})$

$$N(s) = \left(\int_M \nu(s)^2\right)^{1/2}$$

The Yang-Mills Functional

- Then N is a norm equivalent to the usual L^2 norm and so extend to the L^2 cross sections. Now for an L^2_1 connection A define the functional J:

$$J(A) = N(\frac{*F}{2\pi i} + \mu.1),$$

where $\mu = \mu(E)$.

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- **Thus $J(A) = 0$ if and only if the connection is of the type required by the theorem.**
For bundles of rank two and degree zero J is essentially the Yang-Mills functional $||F||_{L^2}$ introduced by Atiyah and Bott.

A Theorem on Convergence of connections

- Proposition(Uhlenbeck, 1981.) Suppose that $A_i \in \mathcal{A}$ is a sequence of L^2 connections with $||F||_{L^2}(A_i)$ bounded. Then there are a subsequence $\{i'\} \subset \{i\}$ and L^2 gauge transformations $u_{i'}$ such that $u_{i'}(A_{i'})$ converges weakly in L^2. The main ingredient for the proof of the following key lemma is the above result by Uhlenbeck.

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- The Key lemma. Let E be a holomorphic bundle over X. Then either $\inf J|_{0(\mathcal{E})}$ is attained in $0(\mathcal{E})$ or there is a holomorphic bundle $\mathcal{F} \ncong \mathcal{E}$ of the same degree and rank as $\mathcal{E}$ and with $\inf J|_{0(\mathcal{F})} < \inf J|_{0(\mathcal{E})}$; $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$.

Sketch of the Proof of the Key Lemma

- Pick a minimizing sequence A_i for $J|_{0(\mathcal{E})}$ Since N is equivalent to the L^2 norm, we have $||F(A)||_{L^2}$ bounded and can apply the Uhlenbeck's theorem to deduce that, $A_i \longrightarrow B$ weakly in L^2 and $J(B) < \liminf J(A_i) = \inf J|_{0(\mathcal{E})}$.

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- To see this, define for any two connections A, A' a connection $d_{AA'}$, on the bundle $\mathrm{Hom}(E, E) = E^* \otimes E$ built from the connection A on the left hand factor and A' on the right, with a corresponding

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- Thus solutions of $\overline{\partial}_{AA'}(s) = 0$ (which exist by ellipticity of $\overline{\partial}$) corresponds exactly to elements of $\mathrm{Hom}(\mathcal{E}_A, \mathcal{E}_{A'})$.

The behavior of Curvature

- Curvature and holomorphic extension.
  The strategy of the proof is that if the bundle $\mathcal{E}$ is stable the second alternative of the key lemma does not occur. In general if $\alpha : \mathcal{E} \to \mathcal{F}$ is a holomorphic map of bundles over M, according to a result by Narasimhan there are proper extensions and a factorization:

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  with rows exact,
- rank($\mathcal{Z}$)= rank($\mathcal{M}$) and $\det\beta \neq 0$, $\deg\mathcal{Z} \leq \deg\mathcal{M}$.

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- Some generalities: If we have any exact sequence of holomorphic bundles $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$ then any unitary connection A on $\mathcal{T}$ has the shape:

$$A = \begin{pmatrix} A_{\mathcal{S}} & \beta \\ -\beta^* & A_{\mathcal{U}} \end{pmatrix}$$

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- where $A_{\mathcal{S}}$, $A_{\mathcal{U}}$ are connections on $\mathcal{S}, \mathcal{U}$ and β in $\Omega^{0,1}(\mathcal{U}^* \otimes \mathcal{S})$. The corresponding curvature matrix has the decomposition:

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- in which the quadratic term have a definite sign. In fact this is a principle that curvature decreases in holomorphic subbundles.

The Behavior of the Yang-Mills Functional

- Lemma 1- If $\mathcal{F}$ is a holomorphic bundle over M which can be expressed as an extension : $0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{N} \to 0$ and if $\mu(\mathcal{M}) \geq \mu(\mathcal{F})$ then for any unitary connection A on $\mathcal{F}$ we have :

$$J(A) \geq \mathrm{rk}\mathcal{M}(\mu(\mathcal{M}) - \mu(\mathcal{F})) + \mathrm{rk}\mathcal{N}(\mu(\mathcal{F}) - \mu(\mathcal{N})) := J_0$$

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- Lemma 2. Suppose that $\mathcal{E}$ is a stable holomorphic bundle and make the inductive hypothesis that the main theorem has been proved for bundles of lower rank. If $\mathcal{E}$ can be expressed as an extension $0 \to \mathcal{P} \to \mathcal{E} \to \mathcal{Z} \to 0$ (so from the definition of stability we have: $\mu(\mathcal{P}) < \mu(\mathcal{E}) < \mu(\mathcal{Z})$), then there is a connection A on $\mathcal{E}$ with

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- The above Lemma is somehow stronger than the Lemma 1 because it is exploiting the special properties of the functional J

Proof of the main theorem

- **Proof of the Main Theorem** According to the inequality in the lemma 1 if $\mathcal{E}$ is a bundle with a connection of the type required by the main theorem i.e. $J = 0$ then $\mathcal{E}$ must be stable.

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- Conversely if $\mathcal{E}$ is stable and the theorem has been proved for bundles of lower ranks then inf $J|_{O(\mathcal{E})}$ is attained in $O(\mathcal{E})$. For if not, the key lemma constructs a bundle $\mathcal{F}$ with $\deg\mathcal{F} = \deg(\mathcal{E}), \mathrm{rank}\mathcal{F} = \mathrm{rank}\mathcal{E}, \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$ and $\inf J|_{O(\mathcal{E})} \geq \inf J|_{O(\mathcal{F})}$.

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- So we can apply lemma 1 to the bottom row of the diagram to deduce

$$\inf J|_{O(\mathcal{F})} \geq J_0$$

  and lemma 2 to the top row to deduce

$$\inf J|_{O(\mathcal{E})} \leq J_1$$

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- implies that $J_1 \leq J_0$ and we obtain

$$\inf J|_{O(\mathcal{E})} \leq J_1 \leq J_0 \leq \inf J|_{O(\mathcal{F})}$$

a contradiction, so $\inf J|_{O(\mathcal{E})}$ is attained in $O(\mathcal{E})$.

Small variation within the Orbit $O(\mathcal{E})$

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- Thus The minimum of the functional J on the orbit $O(\mathcal{E})$ is attained in this orbit , say, at the connection A.
- Now by an infinitesimal argument one can easily show that at the connection A we should have $J(A) = 0$ as it was desired.

Relation to the Lie Theory

- This method can be applied very naturally to the problem of stability of parabolic bundles over marked Riemann surfaces. In Lie theory (or random matrix theory) there was an old problem about determining the spectrum of the product of two fixed conjugacy classes chosen randomly from a compact Lie group, this is the well known "support problem".

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- The surprising fact is that the support problem can be described in term of the stability property of a special complex vector bundle over compact Riemann surface (generally with marked points).

Relation to the Lie Theory

- This relation that for solving a problem in Lie theory we have to go out and use the concept of stability in the realm of differential geometry motivates us to ask whether one can solve the support problem in the Lie group for example $G = SU(n)$ in term of the Lie theory of this group.

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- Question : What is a good counterpart of the notion of stability in differential geometry in the realm of Lie theory.

Thank You for Your Attention